Physical Applications of Harmonic Functions

Siew T. Koay
Department of Electrical and Computer Engineering
Prairie View A&M University

Abstract

An earnest effort has been made to present this tutorial paper with an emphasis on analytical, logical and intuitive thinking; geometric reasoning; and physical description in a way that is more congenial and receptive to the readers, especially engineering students, to fully comprehend the practical power of harmonic functions in solving physical problems in physics and engineering. Mathematically, all these physical problems can be formulated in terms of Laplace's equation. Furthermore, harmonic functions are solutions of Laplace's equation and have continuous second partial derivatives.

This paper commences to give a brief introduction to harmonic functions and proceeds to delve into the general properties of harmonic functions, particularly important are mean value property and maximum modulus property. It then moves on to examine the connection between Laplace's equation and complex analytical functions and illustrate it by working out examples from electrostatics and fluid flow. These examples reveal the unifying power of mathematics: physical problems from electrostatics and fluid flow can be treated by the same mathematical methods. There exists an analogy between them: electrostatic equipotential lines and electrostatic lines of force correspond to the equipotential lines of the velocity potential and the streamlines of fluid flow respectively.

As harmonic functions are the real and imaginary parts of complex analytical function, they remain harmonic under conformal mapping so that conformal mapping becomes a powerful tool in solving boundary value problems. Consequently, conformal mapping can be effectively and efficiently used to solve problems by mapping a given domain onto one for which the solution of the given problem is known or can be solved more easily. The solution thus obtained is then mapped back to the given domain.

Introduction

Harmonic functions, which are solutions of the two-dimensional Laplace's equation having continuous second-order partial derivatives, play a vital role in solving many physical problems, for example, in hydrodynamics, aerodynamics, heat transfer, acoustics, and electrostatics. Mathematically, all these physical problems can be formulated in terms of Laplace's equation:

$$\text{Laplace's equation:} \quad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0, \ \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0$$

where $\phi(x,y)$ and $\varphi(x,y)$ are the real and imaginary parts of analytic function Ω(z).

$$\Omega(z) = \phi + i\varphi$$

The functions ϕ and φ are called conjugate functions; and given one, the other can be determined within an arbitrary additive constant.

It should be noted here that although most harmonic functions do have harmonic conjugates, however this is not always the case. The question as to whether there exists a harmonic conjugate or not can depend on the underlying topology of its domain of definition. In the case where the domain is simply connected, meaning that it contains no holes, then one can always find a harmonic conjugate. Whereas in the other cases where the domains are not simply connected, there may not exist a single-valued harmonic conjugate that can serve as the imaginary part of a well-defined complex function f(z).

It should also be noted here that both functions, $\phi(x,y)$ and $\varphi(x,y)$, remain harmonic under conformal mapping, thereby empowering conformal mapping to model and solve boundary value problems. By conformality, both families of curves, $\phi(x,y) = \text{constant}$ and $\varphi(x,y) = \text{constant}$, are orthogonal.

Basic Properties of Harmonic Functions

Harmonic functions, like analytic functions, have a number of basic properties, particularly important are the mean value property and the maximum modulus property.

1. Mean-Value Properties

Let $\phi(x,y)$ be harmonic in a simply connected domain D. Then the value of $\phi(x,y)$ at a point (x_0, y_0) in D is equal to the mean value of $\phi(x,y)$ on any circle in D with center at (x_0, y_0).

$$\phi(x_0, y_0) = \frac{1}{2\pi}\int_0^{2\pi} \phi(x_0 + r\cos\vartheta, y_0 + r\sin\vartheta)\,d\vartheta$$

which is also equal to the mean value of $\phi(x,y)$ on any circular disk in D with center at (x_0, y_0).

$$\phi(x_0, y_0) = \frac{1}{\pi r_0^2}\int_0^{r_0}\int_0^{2\pi} \phi(x_0 + r\cos\vartheta, y_0 + r\sin\vartheta)\,r\,d\vartheta\,dr$$

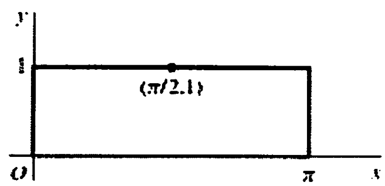
2. Maximum and Minimum Modulus Properties[1]

Let $\phi(x,y)$ be harmonic and not constant in a given domain D, then it has neither a maximum nor a minimum in D. Thus, the maximum and the minimum occur on the boundary C of D.

Corollary (Uniqueness of Harmonic Functions): If a function f(x,y) is harmonic in D and on the boundary C and if f(x,y) = $\phi(x,y)$ on C, then f(x,y) = $\phi(x,y)$ in D as well.

Corollary: If $\phi(x, y)$ has a maximum or a minimum in D, then $\phi(x, y)$ is a constant.

Let us illustrate the Maximum Modulus Principle with an example that shows for f(z) = sin z the maximum value of |f(z)| occurs on the boundary of the rectangular region R: $0 < x < \pi$, $0 < y < 1$ and not in the interior of R.



$$f(z) = \sin z$$

$$|f(z)|^2 = |\sin z|^2 = \sin^2 x + \sinh^2 y$$

$$|f(z)| = \sqrt{\sin^2 x + \sinh^2 y}$$

The term $\sin^2 x$ is greatest when $x = \pi/2$ and that the increasing function $\sinh^2 z$ is greatest when y = 1. Thus, the maximum value of |f(z)| in R occurs at the boundary point z = (π/2, 1) and at no other point in R.

Note: **Uniqueness Theorem for the Dirichlet Problem**: If for a given region and given boundary values, the Dirichlet problem for the Laplace equation in two variables has a unique solution.

The **Dirichlet problem** is the problem of finding a function that is harmonic in a specified domain and the values of the function are prescribed on the boundary of the domain.

3. Zeroes of analytic functions are isolated, whereas zeros of real-valued harmonic functions are never isolated.

4. The conjugate of a given harmonic function is uniquely determined up to an arbitrary real additive constant.

5. Harmonic functions remain harmonic under conformal mapping.

6. If $\emptyset_1, \emptyset_2, \ldots, \emptyset_n$ are harmonic in a region R, and $c_1, c_2, \ldots, c_n$ are any constants, then $c_1\emptyset_1 + c_2\emptyset_2 + \ldots + c_n\emptyset_n$ is harmonic in R.

Physical Application to Fluid Flow

Many physical problems in fluid flow can be solved by complex variable methods under the following basic assumptions:

a) The fluid flow is two dimensional: the basic flow pattern of the fluid motion is assumed to be in the z plane.
b) The flow is stationary. This means that the velocity of the flow at any point depends only on the position (x, y) and not on time. The components of the velocity of the fluid at (x, y) in the positive x and y directions, V_x and V_y, can be derived from a potential function $\emptyset$, called velocity potential, such that

$$V_x = \frac{\partial \phi}{\partial x}, \quad V_y = \frac{\partial \phi}{\partial y} \qquad (1)$$

c) The flow is incompressible ($\nabla \cdot V = 0$, divergence free), irrotational ($\nabla x V = 0$, curl free) and non-viscous. This means that the quantity of fluid entering is equal to the quantity of fluid leaving, the flow is circulation free, and no internal friction.

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = 0 \qquad (2)$$

Substituting equation (1) into equation (2) gives Laplace's equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

thereby the velocity potential ϕ is harmonic. It follows that there must exist a conjugate harmonic function, φ, such that

$$\Omega(z) = \phi(x,y) + i\varphi(x,y)$$

is analytic. The function $\Omega(z)$, of fundamental importance in charactering a flow, is called the complex potential. By differentiation, we have, using Cauchy-Riemann equations and equation (1)

$$\Omega'(z) = \frac{\partial \phi}{\partial x} + i\frac{\partial \varphi}{\partial x} = \frac{\partial \phi}{\partial x} - i\frac{\partial \phi}{\partial y} = V_x - iV_y$$

Thus, the magnitude of the velocity is

$$|V| = \sqrt{V_x^2 + V_y^2} = |\Omega'(z)|$$

Note that points at which $\Omega'(z) = 0$ are called stagnation points.

Example: Flow Around an Obstacle

An interesting and important problem in fluid flow is that of determining the flow pattern of a fluid initially moving with uniform velocity in which an obstacle has been placed. So, let us consider the complex potential of a fluid flow as given below:

$$\Omega(z) = \phi(x,y) + i\varphi(x,y) = V_0(z + \frac{1}{z})$$

Let z = r$e^{i\theta}$. Then

$$\Omega(z) = V_0\left(re^{i\theta} + \frac{1}{r}e^{-i\theta}\right) = V_0\left(r + \frac{1}{r}\right)\cos\theta + iV_0\left(r - \frac{1}{r}\right)\sin\theta$$

from which

$$\phi(x,y) = V_0\left(r + \frac{1}{r}\right)\cos\theta \ \ and\ \ \varphi(x,y) = V_0\left(r - \frac{1}{r}\right)\sin\theta$$

The equipotential lines are given by

$$\phi(x,y) = V_0\left(r + \frac{1}{r}\right)\cos\theta = \text{constant} = \alpha$$

and the streamlines are given by

$$\varphi(x,y) = V_0\left(r - \frac{1}{r}\right)\sin\theta = \text{constant} = \beta$$

Note that $\varphi = \beta = 0$ corresponds to r = 1 and $\theta = 0$ *or* π. The circle r = 1 represents a streamline, and since there cannot be any flow across a streamline, it can be considered as a circular obstacle of radius 1 placed in the path of the fluid.

$$\Omega'(z) = V_0(1 - \frac{1}{z^2}) = V_0(1 - \frac{1}{r^2}e^{-2i\theta}) = V_0(1 - \frac{1}{r^2}\cos 2\theta) + iV_0\frac{1}{r^2}\sin 2\theta$$

Thus,

$$V = \overline{\Omega'(z)} = V_0(1 - \frac{1}{r^2}\cos 2\theta) - iV_0\frac{1}{r^2}\sin 2\theta$$

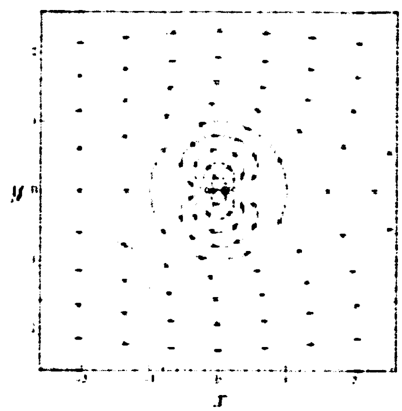
$$|V| = \sqrt{[V_0(1 - \frac{1}{r^2}\cos 2\theta)]^2 + [V_0\frac{1}{r^2}\sin 2\theta]^2}$$

$$= V_0\sqrt{1 - \frac{2}{r^2}\cos 2\theta + \frac{1}{r^4}}$$

which shows that far from the obstacle, the flow is nearly uniform ($|V| \approx V_0$) and parallel to the x-axis.

The flow has two stagnation points (that is, points at which the velocity is zero) at z = ± 1. This follows from $\Omega'(z) = V_0(1 - 1/z^2) = 0$

Notice that there is a singularity at z = 0 which is known as a doublet and corresponds to the function V_0/z. The singularity at the origin is inside the circular obstacle and thus does not affect the external flow. The full streamline pattern, including the doublet inside the circle, is shown below.



Physical Application to Electrostatics

The force of attraction or repulsion between charged particles is governed by Coulomb's law. This vector force induces an electric field. If a unit positive charge, which is small enough so as not to affect the field appreciably, is placed at any point P free of charge, the force acting on this charge is called the electric field intensity at P and is denoted as E. This vector force is the negative gradient of a scalar function Ø, called the electrostatic potential. In symbols,

$$E = -\nabla \emptyset$$

$$E_x = -\frac{\partial \phi}{\partial x},\ E_y = -\frac{\partial \phi}{\partial y}$$

$$E = E_x + iE_y = -\frac{\partial \phi}{\partial x} - i\frac{\partial \phi}{\partial y}$$

It follows that in any region not occupied by charge,

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = 0 \quad \rightarrow \quad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

That is, Ø is harmonic at all points not occupied by charge. Thus, there must exist a conjugate harmonic function $\varphi(x, y)$ such that

$$\Omega(z) = \phi(x, y) + i\,\varphi(x, y)$$

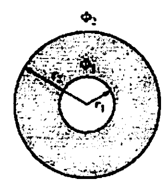
is analytic in any region not occupied by charge. $\Omega(z)$ is called complex electrostatic potential.

$$E = E_x + iE_y = -\frac{\partial \phi}{\partial x} - i\frac{\partial \phi}{\partial y} = -\frac{\partial \phi}{\partial x} + i\frac{\partial \varphi}{\partial x} = -\overline{\Omega'(z)}$$

$$|E| = |\Omega'(z)|$$

The level curves $\phi(x,y) = \alpha$ and $\varphi(x,y) = \beta$, where α and β are constants, are called equipotential lines and flux lines, respectively. Further, if $\Omega'(z) \neq 0$, then $\Omega(z)$ is conformal and the level curves of ϕ and ψ are orthogonal and the electric field intensity vector is tangent to the flux lines.

Example: Potential between concentric cylinders
Consider a region bounded by two infinitely long concentric cylindrical conductors of radii r_1 and r_2 ($r_1 < r_2$), which are charged to potentials ϕ_1 and ϕ_2, respectively



Find (a) the electrostatic potential and (b) electric field vector everywhere in the region.

Solution. (a) By symmetry, the electrostatic potential ϕ depends only on $r = \sqrt{x^2 + y^2}$, and the Laplace's equation in polar coordinates is

$$r^2\phi_{rr} + r\phi_r + \phi_{\theta\theta} = 0.$$

$$r^2\phi_{rr} + r\phi_r = 0 \quad \text{with } \phi_{\theta\theta} = 0.$$

which gives

$$\frac{\phi''}{\phi'} = -\frac{1}{r} \rightarrow \ln\phi' = -\ln r + c \rightarrow \phi' = \frac{\alpha}{r} \rightarrow \phi = \alpha \ln r + \beta$$

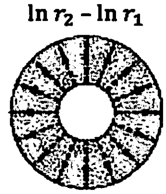
Thus,
$$\phi_1 = \alpha \ln r_1 + \beta, \quad \phi_2 = \alpha \ln r_2 + \beta$$

which give $\alpha = (\phi_2 - \phi_1)/(\ln r_2 - \ln r_1)$ and $\beta = (\phi_1 \ln r_2 - \phi_2 \ln r_1)/(\ln r_2 - \ln r_1)$

Hence
$$\phi = \alpha \ln r + \beta = \frac{(\phi_2 - \phi_1)\ln r + \phi_1 \ln r_2 - \phi_2 \ln r_1}{\ln r_2 - \ln r_1}$$

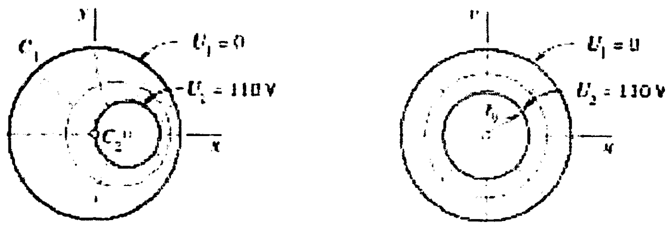
(b) The electric field vector

$$E = -\frac{\partial\phi}{\partial r} = -\frac{(\phi_2 - \phi_1)/r}{\ln r_2 - \ln r_1} = \frac{(\phi_1 - \phi_2)/r}{\ln r_2 - \ln r_1}$$



Note that equipotential lines ϕ = constant are concentric circles centered at the origin and are orthogonal to the lines of electric force or flux lines that are rays emanating from the origin as shown in the figure above.

Example: Potential between non-coaxial cylinders
Consider a region bounded by two infinitely long non-coaxial cylindrical conductors of radii r_1 and r_0 ($r_1 > r_0$), which are charged to potentials $\phi_1 = 0\ V$ and $\phi_2 = 110$ V, respectively[2].



Find the electrostatic potential everywhere in the region.

Solution: First transform the non-coaxial cylinders in the z-plane to coaxial cylinders in the w-plane by the following mapping function

$$w = f(z) = \frac{2z-1}{z-2}$$

Note that the solution to the problem of finding the electrostatic potential between coaxial cylinders can be easily obtained as in the above example. Thus,

$$\phi^* = \alpha \ln|w| + \beta$$

$$\phi_1^* = \alpha \ln r_1 + \beta, \quad \phi_2^* = \alpha \ln r_0 + \beta$$

$$\phi_1^* = 0 = \alpha \ln 1 + \beta \ \rightarrow \ \beta = 0$$

$$\phi_2^* = 110 = \alpha \ln(\tfrac{1}{2}) \ \rightarrow \ \alpha = 110/\ln(1/2) \ = \ -110/0.693 = -158.696$$

$$\phi^* = -158.696 \ln|w|$$

The desired solution in the given domain in the z-plane is

$$\phi = -158.696 \ln\left|\frac{2z-1}{z-2}\right|$$

Note: A knowledge of conformal mapping functions is very useful in solving boundary value problems. The fact that harmonic functions remain harmonic under conformal mapping was utilized to map a given domain onto one for which the solution is known or can be found more easily. This solution is then mapped back to the given domain by employing the inverse mapping function.

Summary and Conclusions

This paper commences to give a brief introduction to harmonic functions, and proceed to delve into the general properties of harmonic functions. It then moves on to illustrate through examples the usefulness of the harmonic functions in solving physical problems in electrostatics and fluid flow. These examples consistently reveal the unifying power of mathematics: physical problems with different phenomena from different areas in physics and engineering having the same type of model can be treated by the same mathematical methods. Furthermore, the fact that harmonic functions remain harmonic under conformal mapping was utilize to solve problems by using conformal mapping to map a given domain onto one for which the solution is known or can be found more easily. This solution is then mapped back to the given domain by employing the inverse mapping function.

References

1. Brown, J. W., Churchill, R. V., Complex Variables and Applications, 8th ed., McGraw-Hill Book Company, New York.
2. Kreyszig, E., Advanced Engineering Mathematics, 10th ed., John Wiley & Sons, Hoboken, New Jersey.

SIEW T. KOAY
Dr. Koay currently serves as Professor in the Department of Electrical and Computer Engineering at Prairie View A&M University, a member of Texas A&M University System. His current research interests include transform coding in communication, stochastic modeling, computer programming, and STEM education.